CENTERS OF DEGENERATE CYCLOTOMIC HECKE ALGEBRAS AND PARABOLIC CATEGORY $\mathcal{O}$

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ABSTRACT. We prove that the center of each degenerate cyclotomic Hecke algebra associated to the complex reflection group of type $B_d(l)$ consists of symmetric polynomials in its commuting generators. The classification of the blocks of the degenerate cyclotomic Hecke algebras is an easy consequence. We then deduce that the center of an integral block of parabolic category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is generated by the center of its universal enveloping algebra.

1. INTRODUCTION

Let R be a fixed commutative ground ring. Recall from [D] that the *degenerate affine Hecke algebra* H_d is the R-algebra which is equal as an R-module to the tensor product $R[x_1, \ldots, x_d] \otimes_R RS_d$ of the polynomial algebra $R[x_1, \ldots, x_d]$ and the group algebra RS_d of the symmetric group S_d. Multiplication is defined so that $R[x_1, \ldots, x_d]$ (identified with the subspace $R[x_1, \ldots, x_d] \otimes 1$) and RS_d (identified with the subspace $1 \otimes RS_d$) are subalgebras, and in addition

$$s_i x_{i+1} = x_i s_i + 1,$$
$$s_i x_j = x_j s_i \qquad (j \neq i, i+1),$$

where s_i denotes the basic transposition $(i\, i+1) \in S_d$. It is known by [L, Theorem 6.5] that the center $Z(H_d)$ of H_d consists of all symmetric polynomials in the (algebraically independent) generators $x_1, \ldots, x_d$.

Given in addition a monic polynomial $f(x) = x^l + c_1 x^{l-1} + \cdots + c_l \in R[x]$ of degree $l \geq 1$, the *degenerate cyclotomic Hecke algebra* H_d^f is the quotient of H_d by the two-sided ideal generated by $f(x_1)$. We refer to l here as the *level*. Since we seldom mention H_d itself again, it should not cause confusion to also use the notation $x_1, \ldots, x_d$ for the canonical images of the polynomial generators of H_d in the quotient H_d^f. For example, if $f(x) = x$ then H_d^f can be identified simply with the group algebra RS_d, and under this identification we have that

$$x_i = \sum_{j=1}^{i-1} (j\, i) \ \in RS_d,$$

the *Jucys-Murphy elements*. In this case, it has long been known (see [J] or [M, 1.9]) that the center of RS_d again consists of all symmetric polynomials in $x_1, \ldots, x_d$, though of course these generators are no longer algebraically independent. In other words, the canonical homomorphism $H_d \twoheadrightarrow RS_d$ maps $Z(H_d)$ surjectively onto $Z(RS_d)$. Our first result proves the analogous statement for the quotient map $H_d \twoheadrightarrow H_d^f$ in general.

Theorem 1. *The center of H_d^f consists of all symmetric polynomials in $x_1, \ldots, x_d$. Moreover, $Z(H_d^f)$ is free as an R-module with an explicit basis parametrized by all l-multipartitions of d.*

For the first application, specialize to the case that $R = F$ is an algebraically closed field. We say that two irreducible modules L and L' *belong to the same block* if they are linked by a chain $L = L_0, L_1, \ldots, L_n = L'$ of irreducible modules such that there is a non-split extension between L_{i-1} and L_i for each $i = 1, \ldots, n$. For modules over a finite dimensional algebra like H_d^f, this is equivalent to the property that L and L' have the same central character. So, on combining Theorem 1 with the existing theory, we obtain the classification of the blocks of the degenerate cyclotomic Hecke algebras.[1] The conclusion is exactly as claimed in Grojnowski's unpublished note [G]; see §4 below for the precise statement. Unfortunately, as has been pointed out by Anton Cox, the argument given there is incomplete, so this corrects an error in the literature. Actually, [G] was mainly concerned with cyclotomic Hecke algebras (not their rational degenerations). For these, it has also long been expected that the center consists of all symmetric polynomials in the Jucys-Murphy elements, but we still do not know how to prove this. Nevertheless, Lyle and Mathas [LM] have recently managed to solve the problem of classifying the blocks of the cyclotomic Hecke algebras too, by a quite different method.

Now we further specialize to the case that $F = \mathbb{C}$. Let $\mu = (\mu_1, \ldots, \mu_l)$ be an l-tuple of positive integers summing to n. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and let $\mathfrak{p}$ be the standard parabolic subalgebra with block diagonal Levi subalgebra $\mathfrak{h} = \mathfrak{gl}_{\mu_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{\mu_l}(\mathbb{C})$. Let $\mathcal{O}^\mu$ be the category of all *finitely generated* $\mathfrak{g}$-modules which are *locally finite* as $\mathfrak{p}$-modules and *integrable* as $\mathfrak{h}$-modules, i.e. they lift to rational representations of $H = GL_{\mu_1}(\mathbb{C}) \times \cdots \times GL_{\mu_l}(\mathbb{C})$. This is the usual parabolic analogue of the BGG category $\mathcal{O}$, except that we are only allowing modules with integral weights/central characters. The category $\mathcal{O}^\mu$ decomposes as

$$\mathcal{O}^\mu = \bigoplus_\nu \mathcal{O}^\mu_\nu$$

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[1] In an earlier version of this article, we also explained how to deduce the classification of blocks of the degenerate affine Hecke algebra H_d from Theorem 1. However, Iain Gordon has pointed out that this follows immediately by a general result of Müller [BG, III.9.2], since H_d is finite as a module over its center.

where the direct sum is over integral central characters $\nu : Z(\mathfrak{g}) \to \mathbb{C}$ of the universal enveloping algebra $U(\mathfrak{g})$, and $\mathcal{O}_\nu^\mu$ is the full subcategory of $\mathcal{O}^\mu$ consisting of modules with generalized central character ν.

The next result, also ultimately a consequence of Theorem 1, is an essential ingredient in [B2, S], which give quite different (and independent) proofs of a conjecture of Khovanov [Kh, Conjecture 3]. Recall that the center $Z(\mathcal{C})$ of an additive category $\mathcal{C}$ is the commutative ring consisting of all natural transformations from the identity functor to itself. For example, if $\mathcal{C}$ is the category of finite dimensional modules over a finite dimensional algebra C, then $Z(\mathcal{C})$ is canonically isomorphic to the center of the algebra C itself.

Theorem 2. *For any integral central character ν, the natural map*

$$m_\nu^\mu : Z(\mathfrak{g}) \to Z(\mathcal{O}_\nu^\mu)$$

sending $z \in Z(\mathfrak{g})$ to the natural transformation defined by left multiplication by z is a surjective algebra homomorphism. Moreover, the dimension of $Z(\mathcal{O}_\nu^\mu)$ is the same as the number of isomorphism classes of irreducible modules in $\mathcal{O}_\nu^\mu$.

The category $\mathcal{O}_\nu^\mu$ is equivalent to the category of finite dimensional modules over a finite dimensional algebra, e.g. one can take endomorphism algebra of a minimal projective generator. Hence, two irreducible modules in $\mathcal{O}_\nu^\mu$ belong to the same block if and only if they have the same central character with respect to $Z(\mathcal{O}_\nu^\mu)$. By definition, all irreducible modules in $\mathcal{O}_\nu^\mu$ have the same central character with respect to $Z(\mathfrak{g})$. So Theorem 2 implies that all the irreducible modules in $\mathcal{O}_\nu^\mu$ belong to the same block. This proves that the above decomposition of $\mathcal{O}^\mu$ (defined by central characters) coincides with its decomposition into blocks in the usual sense (defined by linkage classes of irreducible modules). For regular central characters in arbitrary type, this is a known consequence of some results of Deodhar combined with the Kazhan-Lusztig conjecture, but for singular central characters even in type A this was an open problem. According to Boe, the same coincidence is expected in types D and E, but there are counterexamples in non-simply-laced types.

The remainder of the article is organized as follows. There is a natural filtration on the algebra H_d^f with respect to which the associated graded algebra $\operatorname{gr} H_d^f$ is the twisted tensor product of the level l truncated polynomial algebra $R[x_1, \ldots, x_d]/(x_1^l, \ldots, x_d^l)$ by the group algebra RS_d of the symmetric group. In section 2, we compute the center of this associated graded algebra directly, giving the crucial upper bound on the size of $Z(H_d^f)$ since we obviously have that $\operatorname{gr} Z(H_d^f) \subseteq Z(\operatorname{gr} H_d^f)$. There are then several different ways to show that this upper bound is actually attained. The approach followed in section 3 is to simply write down enough linearly independent central elements in H_d^f. This has the advantage of yielding at the same time an explicit basis for $Z(H_d^f)$ which is a generalization of Murphy's basis

for $Z(RS_n)$ constructed in the proof of [M, 1.9]. In section 4, we discuss the classification of the blocks of H_d^f in more detail. In particular we compute the dimension of the center of each block, refining Theorem 1 which gives the dimension of the center of the whole algebra. Finally in section 5 we deduce the results about parabolic category $\mathcal{O}$ by exploiting the Schur-Weyl duality for higher levels from [BK], which reduces many questions about the category $\mathcal{O}^\mu$ to the degenerate cyclotomic Hecke algebras H_d^f for $f(x) = (x - \mu_1) \cdots (x - \mu_l)$.

2. THE CENTER OF THE ASSOCIATED GRADED ALGEBRA

We fix an integer $l \geq 1$ and a commutative ring R. Let $R_l[x_1, \ldots, x_d]$ denote the level l truncated polynomial algebra, that is, the quotient of the polynomial algebra $R[x_1, \ldots, x_d]$ by the relations $x_1^l = \cdots = x_d^l = 0$. The symmetric group S_d acts on $R_l[x_1, \ldots, x_d]$ by algebra automorphisms so that $w \cdot x_i = x_{wi}$ for each i and $w \in S_d$. We view the resulting twisted tensor product algebra $R_l[x_1, \ldots, x_d] \otimes RS_d$ as a graded algebra with each x_i in degree 1 and all elements of S_d in degree 0. The goal in this section is to compute the center of this algebra explicitly. We remark that the algebra $R_l[x_1, \ldots, x_d] \otimes RS_d$ can be viewed as a degeneration of the group algebra $R(C_l \wr S_d)$ of the wreath product of the symmetric group and the cyclic group of order l. It is well known that the conjugacy classes of $C_l \wr S_d$ are parametrized by certain multipartitions; see [Mac, p.170] or [W]. With this in mind the results in this section should not be too surprising.

Let Q_d denote the centralizer of $R_l[x_1, \ldots, x_d]$ in $R_l[x_1, \ldots, x_d] \otimes RS_d$. The symmetric group S_d acts on Q_d by conjugation, i.e. $w \cdot z = wzw^{-1}$. It is obvious that the center of $R_l[x_1, \ldots, x_d] \otimes RS_d$ is just the set of fixed points:

$$Z(R_l[x_1, \ldots, x_d] \otimes RS_d) = Q_d^{S_d}.$$

We are going first to describe an explicit basis for Q_d from which it will be easy to determine the S_d-fixed points, hence the center.

For $r \geq 0$ and any set $I = \{i_1, \ldots, i_a\}$ of a distinct numbers chosen from $\{1, \ldots, d\}$, let

$$h_r(I) = h_r(i_1, \ldots, i_a) := \sum_{\substack{0 \leq r_1, \ldots, r_a < l \\ r_1 + \cdots + r_a = (a-1)(l-1) + r}} x_{i_1}^{r_1} \cdots x_{i_a}^{r_a} \in R_l[x_1, \ldots, x_d],$$

the $((a - 1)(l - 1) + r)$th complete symmetric function in the variables $x_{i_1}, \ldots, x_{i_a}$. By the pigeonhole principle, $h_r(I)$ is zero if $r \geq l$, and moreover $h_{l-1}(I) = x_{i_1}^{l-1} \cdots x_{i_a}^{l-1}$.

Lemma 2.1. *Let I, J be any two subsets of $\{1, \ldots, d\}$ with $c = |I \cap J| > 0$. For any $r, s \geq 0$, we have that $h_r(I)h_s(J) = l^{c-1}h_{r+s+(c-1)(l-1)}(I \cup J)$.*

Proof. Suppose first that $I = \{i_1, \ldots, i_a, k\}$ and $J = \{j_1, \ldots, j_b, k\}$ with $I \cap J = \{k\}$. Then we have that

$$h_r(I)h_s(J) = \sum_{\substack{r_1, \ldots, r_a \\ s_1, \ldots, s_b}} x_{i_1}^{r_1} \cdots x_{i_a}^{r_a} x_k^{a(l-1)+r-r_1-\cdots-r_a} x_{j_1}^{s_1} \cdots x_{j_b}^{s_b} x_k^{b(l-1)+s-s_1-\cdots-s_b}$$

$$= \sum_{\substack{r_1, \ldots, r_a \\ s_1, \ldots, s_b}} x_{i_1}^{r_1} \cdots x_{i_a}^{r_a} x_{j_1}^{s_1} \cdots x_{j_b}^{s_b} x_k^{(a+b)(l-1)+r+s-r_1-\cdots-r_a-s_1-\cdots-s_b}$$

$$= h_{r+s}(I \cup J).$$

This proves the lemma in the case $c = 1$. Next we take $i \neq j$ and note that

$$h_0(i,j)h_0(i,j) = \sum_{r,s} x_i^r x_j^{l-1-r} x_j^s x_i^{l-1-s}$$

$$= \sum_{r,s} x_i^{l-1+r-s} x_j^{l-1+s-r} = l x_i^{l-1} x_j^{l-1} = l h_{l-1}(i,j).$$

Finally take $I = \{i_1, \ldots, i_a, k_1, \ldots, k_c\}$, $J = \{j_1, \ldots, j_b, k_1, \ldots, k_c\}$ for $c \geq 2$ and assume that $I \cap J = \{k_1, \ldots, k_c\}$. Using the preceeding two formulae, we get that

$$h_r(I)h_s(J) = h_r(i_1, \ldots, i_a, k_1)h_0(k_1, k_2) \cdots h_0(k_{c-1}, k_c)$$

$$\times h_0(k_1, k_2) \cdots h_0(k_{c-1}, k_c)h_s(j_1, \ldots, j_b, k_1)$$

$$= l^{c-1} h_{r+s+(c-1)(l-1)}(i_1, \ldots, i_a, j_1, \ldots, j_b, k_1, \ldots, k_c)$$

$$= l^{c-1} h_{r+s+(c-1)(l-1)}(I \cup J).$$

This is what we wanted. $\square$

Now let $A = (i_1 \cdots i_a)$ be an a-cycle in S_d. Write $h_r(A)$ for $h_r(i_1, \ldots, i_a)$. Given another cycle $B = (j_1 \cdots j_b)$, write $A \cup B$ and $A \cap B$ for the *sets* $\{i_1, \ldots, i_a\} \cup \{j_1, \ldots, j_b\}$ and $\{i_1, \ldots, i_a\} \cap \{j_1, \ldots, j_b\}$, respectively. Let

$$A^{(r)} := h_r(A)A \in R_l[x_1, \ldots, x_d] \otimes RS_d,$$

which we call a cycle of *color* r. As before, we have that $A^{(r)} = 0$ for $r \geq l$, so we need only consider colors from the set $\{0, 1, \ldots, l-1\}$. In the case of 1-cycles, we have that $(i)^{(r)} = x_i^r$, so 1-cycles of color 0 are trivial.

Lemma 2.2. *Let $r, s \geq 0$ be colors and A and B be cycles in S_d. Let $c = |A \cap B|$.*

   (i) *If $c = 0$ then $A^{(r)}B^{(s)} = B^{(s)}A^{(r)}$, i.e. disjoint colored cycles commute.*

   (ii) *If $c = 1$ (in which case the product AB is a single cycle) then*

$$A^{(r)}B^{(s)} = (AB)^{(r+s)}.$$

   (iii) *If $c \geq 2$ then*

$$A^{(r)}B^{(s)} = \delta_{r+s,0} l^{c-1} h_{(c-1)(l-1)}(A \cup B)AB.$$

Proof.   (i) Obvious.

(ii) Say $A = (i_1 \cdots i_a\, k)$ and $B = (j_1 \cdots j_b\, k)$. By Lemma 2.1, we have that

$$
\begin{aligned}
A^{(r)} B^{(s)} &= h_r(i_1, \ldots, i_a, k) A h_s(j_1, \ldots, j_b, k) B \\
&= h_r(i_1, \ldots, i_a, k) h_s(j_1, \ldots, j_b, i_1) AB \\
&= h_{r+s}(i_1, \ldots, i_a, j_1, \ldots, j_b, k) AB = (AB)^{(r+s)}
\end{aligned}
$$

as required.

(iii) Arguing exactly as in (ii), we get that

$$
A^{(r)} B^{(s)} = l^{c-1} h_{r+s+(c-1)(l-1)}(A \cup B) AB.
$$

Now observe that $h_{r+s+(c-1)(l-1)}(A \cup B)$ is zero unless $r + s = 0$. □

Now we are going to consider products of colored cycles. Using Lemma 2.2, it is easy to see that any such product is either zero or else it can be rewritten as some power of l times a *product of disjoint colored cycles*, meaning a product $A_1^{(r_1)} \cdots A_m^{(r_m)}$ where $A_1, \ldots, A_m$ are disjoint cycles in S_d and $0 \le r_1, \ldots, r_m < l$ are some colors. Moreover, two such products of disjoint colored cycles are equal if and only if one can be obtained from the other by reordering the disjoint colored cycles and adding/removing some 1-cycles of color 0. For example,

$$
(1\,2\,3)^{(4)}(7\,9\,2)^{(1)} = ((1\,2\,3)(7\,9\,2))^{(5)} = (1\,2\,7\,9\,3)^{(5)} = (1\,2\,7\,9\,3)^{(5)}(4)^{(0)},
$$

$$
(1\,2\,3)^{(4)}(7\,9\,2\,1)^{(1)} = 0 = (1\,2\,3)^{(0)}(7\,9\,3\,2\,1)^{(0)} \text{ (assuming } l > 1),
$$

$$
(1\,2\,3)^{(0)}(7\,9\,2\,1)^{(0)} = l(x_1 x_2 x_3 x_7 x_9)^{l-1}(1\,2\,3)(7\,9\,2\,1) = l(1\,7\,9\,3)^{(l-1)}(2)^{(l-1)}.
$$

Theorem 2.3. *The set of all products of disjoint colored cycles is a basis for Q_d. In particular, it is a free R-module of rank*

$$
\sum \frac{d!}{r_1! r_2! \cdots} \left(\frac{l}{1}\right)^{r_1} \left(\frac{l}{2}\right)^{r_2} \cdots
$$

summing over all partitions $(1^{r_1} 2^{r_2} \cdots)$ of d.

Proof.   Observe by applying Lemma 2.2 with $B^{(s)} = (i)^{(1)} = x_i$ that every colored cycle $A^{(r)}$ belongs to the algebra Q_d. Hence all products of colored cycles belong to Q_d. Moreover, any product of colored cycles is a linear combination of products of disjoint colored cycles, and the set of all products of disjoint colored cycles is linearly independent. It just remains to show that Q_d is spanned by all products of colored cycles.

Suppose to start with that $A = (i_1 \cdots i_a)$ is an a-cycle in S_d and that

$$
z = \sum_{\substack{0 \le r_1, \ldots, r_a < l \\ r_1 + \cdots + r_a = k}} c_{r_1, \ldots, r_a} x_{i_1}^{r_1} \cdots x_{i_a}^{r_a} A
$$

is a non-zero homogeneous element of Q_d of degree $k \ge 0$ for some coefficients $c_{r_1, \ldots, r_a} \in R$. We claim that z is a scalar multiple of $A^{(r)}$ for some $0 \le r < l$

(in which case $k = (a-1)(l-1)+r$). To see this, equating coefficients of $x_{i_1}^{r_1} \cdots x_{i_j}^{r_j+1} \cdots x_{i_a}^{r_a} A$ in the equation $x_{i_j} z = z x_{i_j}$ gives that

$$c_{r_1,\ldots,r_j,r_{j+1},\ldots,r_a} = c_{r_1,\ldots,r_j+1,r_{j+1}-1,\ldots,r_a}$$

whenever $r_j < l-1$ for some $j = 1, \ldots, a-1$, interpreting the right hand side as zero in case $r_{j+1} = 0$. If $k = (a-1)(l-1)+r$ for $0 \le r < l$, we deduce from this that all the coefficents $c_{r_1,\ldots,r_a}$ are equal to $c_{l-1,\ldots,l-1,r}$, hence z is a scalar multiple of $A^{(r)}$. Otherwise, we can write $k = m(l-1)+r$ for some $m \le a-2$ and $0 \le r < l-1$ and get that all the coefficients $c_{r_1,\ldots,r_a}$ are equal to $c_{l-1,\ldots,l-1,r,0,0,\ldots,0} = c_{l-1,\ldots,l-1,r+1,-1,0,\ldots,0} = 0$, contradicting the assumption that $z \ne 0$.

Now take an element $fw \in Q_d$ for $w \in S_d$ and a homogeneous polynomial $f \in R_l[x_1, \ldots, x_d]$. Write $w = A_1 \cdots A_m$ as a product of disjoint cycles, none of which are 1-cycles. We show by induction on m that fw is a linear combination of products of colored cycles. The base case $m = 0$ is clear as then $w = 1$. For the induction step, suppose that $m \ge 1$ and $A_m = (i_1 \cdots i_a)$. Let $I = \{i_1, \ldots, i_a\}$ and $J = \{1, \ldots, d\} \setminus I$. We can write $f = \sum_{s=1}^t f_s g_s$ for homogeneous polynomials $f_1, \ldots, f_t \in R_l[x_i \mid i \in I]$ and linearly independent homogeneous polynomials $g_1, \ldots, g_t \in R_l[x_j \mid j \in J]$. Equating coefficients of $g_s A_1 \cdots A_{m-1}$ in the equations $x_i fw = fwx_i$ for each $i \in I$, we deduce that each $f_s A_m$ belongs to Q_d. Hence by the previous paragraph each $f_s A_m$ is a scalar multiple of $A_m^{(r)}$ for some $0 \le r < l-1$. This shows that $fw = \sum_{r=0}^{l-1} h_r A_1 \cdots A_{m-1} A_m^{(r)}$ for homogeneous polynomials $h_r \in R_l[x_j \mid j \in J]$. Equating coefficients of $A_m^{(r)}$ in the equations $x_j fw = fwx_j$ for each $j \in J$, we deduce that each $h_r A_1 \cdots A_{m-1}$ belongs to Q_d. Hence by the induction hypothesis each $h_r A_1 \cdots A_{m-1}$ is a linear combination of products of colored cycles. Hence fw is too.

Finally take an arbitrary homogeneous element $\sum_{w \in S_d} f_w w \in Q_d$, for polynomials $f_w \in R_l[x_1 \ldots, x_d]$. We have for each i that $\sum_{w \in S_d} x_i f_w w = \sum_{w \in S_d} x_{wi} f_w w$. Equating coefficients gives that $x_i f_w w = x_{wi} f_w w = f_w w x_i$ for each i and w. Hence each $f_w w$ belongs to Q_d. So by the previous paragraph each $f_w w$ is a linear combination of products of colored cycles. This completes the proof. $\square$

For a partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ we write $|\lambda|$ for $\lambda_1 + \lambda_2 + \cdots$ and $\ell(\lambda)$ for its *length*, that is, the number of non-zero parts. By an *l-multipartition of d* we mean a tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(l)}| = d$. Let $\mathcal{M}_d(l)$ denote the set of all l-multipartitions of d. Given a product $z = A_1^{(r_1)} \cdots A_m^{(r_m)}$ of disjoint colored cycles in Q_d, where each A_i is an a_i-cycle, we can add extra 1-cycles of color 0 if necessary to assume that $a_1 + \cdots + a_m = d$. Define the *cycle type* of z to be the l-multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ of d defined by declaring that $\lambda^{(r)}$ is the partition whose parts consist of all the a_i such that $r_i = r - 1$. For

$\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)}) \in \mathcal{M}_d(l)$, let $z_d(\boldsymbol{\lambda})$ denote the sum of all products of disjoint colored cycles in Q_d of cycle type $\boldsymbol{\lambda}$.

Theorem 2.4. *The elements $\{z_d(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathcal{M}_d(l)\}$ form a basis for the center of $R_l[x_1, \ldots, x_d] \otimes RS_d$. In particular, $Z(R_l[x_1, \ldots, x_d] \otimes RS_d)$ is a free R-module of rank $|\mathcal{M}_d(l)|$.*

Proof. As we remarked at the beginning of the section, the center of $R_l[x_1, \ldots, x_d] \otimes RS_d$ is the set of fixed points of S_d on Q_d. Given a colored cycle $A^{(r)} = (i_1 \cdots i_a)$ and $w \in S_d$, we have that

$$w \cdot A^{(r)} = (w \cdot A)^{(r)} = (wi_1 \cdots wi_a)^{(r)}.$$

So the action of S_d on Q_d is the linear action induced by a permutation action on the basis from Theorem 2.3. It just remains to observe that two products of disjoint colored cycles lie in the same S_d-orbit if and only if they have the same cycle type, and the $z_d(\boldsymbol{\lambda})$'s are simply the orbit sums. □

Corollary 2.5. *If $d!$ is invertible in R then the center of $R_l[x_1, \ldots, x_d] \otimes RS_d$ is generated by the elements*

$$z_d(a^{(r)}) := \sum_{all\ a\text{-}cycles\ A \in S_d} A^{(r)}$$

for all $0 \leq r < l$ and $1 \leq a \leq d$.

Proof. Take a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)}) \in \mathcal{M}_d(l)$. Consider the product of the elements $z_d(a^{(r-1)})$ over all $r = 1, \ldots, l$ and all non-zero parts a of $\lambda^{(r)}$. It gives an invertible scalar multiple of $z_d(\boldsymbol{\lambda})$ modulo lower terms. □

In the remainder of the section, we are going to construct another basis for $Z(R_l[x_1, \ldots, x_d] \otimes RS_d)$ which is a generalization of the basis for the center of RS_n constructed by Murphy in the proof of [M, 1.9]. Given $k \geq 0$ and $1 \leq i \leq d$, write $k = (a - 1)l + r$ for $a \geq 1$ and $0 \leq r < l$, then set

$$y_i(k) := \sum_{\substack{1 \leq i_1, \ldots, i_{a-1} < i \\ i_1, \ldots, i_{a-1}\ \text{distinct}}} (i_1 \cdots i_{a-1}\ i)^{(r)},$$

an element of degree $(a - 1)(l - 1) + r$. For example, $y_i(k) = 0$ if $k \geq il$ and $y_i(r) = (i)^{(r)} = x_i^r$ for $0 \leq r < l$. Particularly important, we have that

$$y_i(l) = \sum_{j=1}^{i-1} (j\ i)^{(0)},$$

which we call the ith *colored Jucys-Murphy element*.

Lemma 2.6. *For any $1 \leq i \leq d$ and $p \geq 0$, we have that*

$$y_i(l)^p = y_i(pl) + (*)$$

where $(*)$ is a linear combination of products $A_1^{(l-1)} \cdots A_m^{(l-1)}$ for disjoint cycles $A_1, \ldots, A_m$ in S_i such that A_1 involves i and $|A_1 \cup \cdots \cup A_m| \leq p$.

Proof.   Induction exercise using Lemma 2.2.   ☐

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, let $\lambda/l := (\lfloor \lambda_1/l \rfloor \geq \lfloor \lambda_2/l \rfloor \geq \cdots)$. We are going to use partitions belonging to the set

$$\mathcal{P}_d(l) = \{\lambda \mid \ell(\lambda) + |\lambda/l| \leq d\}$$

to parametrize our new basis. Note to start with that $|\mathcal{P}_d(l)| = |\mathcal{M}_d(l)|$, so this set is of the right size. Indeed, there is a bijection

$$\varphi : \mathcal{M}_d(l) \to \mathcal{P}_d(l),$$

defined as follows. Suppose that $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)}) \in \mathcal{M}_d(l)$ where $\lambda^{(r)} = (\lambda_1^{(r)} \geq \cdots \geq \lambda_{m_r}^{(r)} > 0)$. Then $\varphi(\boldsymbol{\lambda})$ denotes the ordinary partition with parts $(\lambda_i^{(r)} - 1)l + r - 1$ for all $1 \leq r \leq l$ and $1 \leq i \leq m_r$. It is easy to see that $\varphi(\boldsymbol{\lambda})$ belongs to $\mathcal{P}_d(l)$. Conversely, given $\mu = (\mu_1 \geq \mu_2 \geq \cdots) \in \mathcal{P}_d(l)$, there is a unique multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ of d such that $\varphi(\boldsymbol{\lambda}) = \mu$: the parts of $\lambda^{(r)}$ are the numbers $\lfloor \mu_i/l \rfloor + 1$ for all $i = 1, \ldots, d - |\mu/l|$ such that $\mu_i \equiv r - 1 \pmod{l}$. Hence φ is indeed a bijection.

Since every element μ of $\mathcal{P}_d(l)$ is of length at most d, it can be thought of simply as a d-tuple of integers. Given two d-tuples $\mu = (\mu_1, \ldots, \mu_d)$ and $\nu = (\nu_1, \ldots, \nu_d)$ we write $\mu \sim \nu$ if one is obtained from the other by permuting the entries. For $\mu = (\mu_1, \ldots, \mu_d) \in \mathcal{P}_d(l)$, define

$$m_d(\mu) := \sum_{\nu \sim \mu} y_1(\nu_1) \cdots y_d(\nu_d).$$

This is a homogeneous element of Q_d of degree $|\mu| - |\mu/l|$.

Theorem 2.7. *The elements $\{m_d(\mu) \mid \mu \in \mathcal{P}_d(l)\}$ form a basis for the center of $R_l[x_1, \ldots, x_d] \otimes RS_d$.*

Proof.   Let us first check that $m_d(\mu)$ belongs to $Z(R_l[x_1, \ldots, x_d] \otimes RS_d)$. We just need to check it commutes with each basic transposition $(i\ i+1)$. Obviously, $(i\ i+1)$ commutes with $y_j(m)$ if $j \neq i, i+1$. Therefore it suffices to show for each $i = 1, \ldots, d-1$ and $k, m \geq 0$ that $(i\ i+1)$ commutes with both the elements $y_i(k)y_{i+1}(k)$ and $y_i(k)y_{i+1}(m) + y_i(m)y_{i+1}(k)$. For the first case, write $k = (a-1)l + r$ as usual. We have that

$$y_i(k)y_{i+1}(k) = \sum_{\substack{1 \leq i_1, \ldots, i_{a-1} < i \\ 1 \leq j_1, \ldots, j_{a-1} < i+1}} (i_1 \cdots i_{a-1}\ i)^{(r)}(j_1 \cdots j_{a-1}\ i+1)^{(r)}$$

where the sum is over distinct $i_1, \ldots, i_{a-1}$ and distinct $j_1, \ldots, j_{a-1}$. We split this sum into two pieces:

$$\sum_{\substack{1 \leq i_1, \ldots, i_{a-1} < i \\ 1 \leq j_1, \ldots, j_{a-1} < i}} (i_1 \cdots i_{a-1}\ i)^{(r)}(j_1 \cdots j_{a-1}\ i+1)^{(r)}$$

which clearly commutes with $(i\,i{+}1)$, and

$$\sum_{b=1}^{a-1} \sum_{\substack{1\le i_1,\ldots,i_{a-1}<i \\ 1\le j_1,\ldots,j_{a-1}<i+1 \\ j_b=i}} (i_1 \,\cdots\, i_{a-1}\, i)^{(r)}(j_1 \,\cdots\, j_{b-1}\, i\, j_{b+1} \,\cdots\, j_{a-1}\, i{+}1)^{(r)}$$

which also commutes with $(i\,i{+}1)$ by an application of Lemma 2.2. The second case is similar.

Now we compare the $m_d(\mu)$'s with the basis from Theorem 2.4. For any $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$ define $\#\boldsymbol{\lambda}$ to be $(d{-}z)$ where z is the number of parts of $\lambda^{(1)}$ that equal 1. We claim for $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$ with $\varphi(\boldsymbol{\lambda}) = \mu$ that $m_d(\mu) = z_d(\boldsymbol{\lambda}) + (*)$ where $(*)$ is a linear combination of $z_d(\boldsymbol{\nu})$'s for $\boldsymbol{\nu} \in \mathcal{M}_d(l)$ with $\#\boldsymbol{\nu} < \#\boldsymbol{\lambda}$. The theorem clearly follows from this claim and Theorem 2.4. To prove the claim, let $\mu = (\mu_1 \ge \cdots \ge \mu_h > 0)$ and write each μ_i as $(a_i - 1)l + r_i$ as usual, so $\#\boldsymbol{\lambda} = a_1 + \cdots + a_h$. By definition, $m_d(\mu)$ is a sum of products of colored cycles of the form $z = A_1^{(r_1)} \cdots A_h^{(r_h)}$ where each A_i is an a_i-cycle. If $A_1, \ldots, A_h$ happen to be disjoint cycles then z is of cycle type $\boldsymbol{\lambda}$. Otherwise, $|A_1 \cup \cdots \cup A_h| < \#\boldsymbol{\lambda}$ so using Lemma 2.2 we can rewrite z as a linear combination of products of disjoint colored cycles of cycle type $\boldsymbol{\nu} \in \mathcal{M}_d(l)$ with $\#\boldsymbol{\nu} < \#\boldsymbol{\lambda}$. Combined with the first paragraph and Theorem 2.4, this shows that $m_d(\mu) = cz_d(\boldsymbol{\lambda}) + (*)$ for some c. Finally, to show that $c = 1$, consider the coefficient of one particular product of disjoint colored cycles of cycle type $\boldsymbol{\lambda}$ in the expansion of $m_d(\mu)$. $\quad\square$

3. The center of H_d^f

We are ready to tackle the problem of computing the center of the degenerate cyclotomic Hecke algebra H_d^f, where $f(x) = x^l + c_1 x^{l-1} + \cdots + c_l \in R[x]$ is a monic polynomial of degree l. Define a filtration

$$\mathrm{F}_0\, H_d^f \subseteq \mathrm{F}_1\, H_d^f \subseteq \mathrm{F}_2\, H_d^f \subseteq \cdots$$

of the algebra H_d^f by declaring that $\mathrm{F}_r\, H_d^f$ is spanned by all $x_{i_1} \cdots x_{i_s} w$ for $0 \le s \le r, 1 \le i_1,\ldots,i_s \le d$ and $w \in S_d$. So each x_i is in filtered degree 1 and each $w \in S_d$ is in filtered degree 0. Given an element $z \in \mathrm{F}_r\, H_d^f$, we write $\mathrm{gr}_r z$ for its canonical image in the rth graded component $\mathrm{gr}_r H_d^f = \mathrm{F}_r\, H_d^f / \mathrm{F}_{r-1}\, H_d^f$ of the associated graded algebra $\mathrm{gr}\, H_d^f = \bigoplus_{r\ge 0} \mathrm{gr}_r H_d^f$. By the PBW theorem for degenerate cyclotomic Hecke algebras [BK, Lemma 3.5], this associated graded algebra $\mathrm{gr}\, H_d^f$ can be identified with the twisted tensor product $R_l[x_1,\ldots,x_d] \otimes RS_d$ so that $\mathrm{gr}_1 x_i$ is identified with $x_i \in R_l[x_1,\ldots,x_d] \otimes RS_d$ and $\mathrm{gr}_0 w$ is identified with $w \in R_l[x_1,\ldots,x_d] \otimes RS_d$. To avoid confusion, we reserve the notations x_i^r and s_i from now on for the elements of H_d^f, always using the alternate notations $(i)^{(r)}$ and $(i\,i{+}1)$ for the corresponding elements of the associated graded algebra $R_l[x_1,\ldots,x_d] \otimes RS_d$.

Given an R-submodule V of H_d^f, we can consider the *induced filtration* on V defined by setting $\mathrm{F}_r\, V := V \cap \mathrm{F}_r\, H_d^f$. The associated graded module $\operatorname{gr} V$ is canonically identified with an R-submodule of $R_l[x_1, \ldots, x_d] \otimes RS_d$, and for two submodules we have that $V = V'$ if and only if $\operatorname{gr} V = \operatorname{gr} V'$. Note also that

$$\operatorname{gr} Z(H_d^f) \subseteq Z(R_l[x_1, \ldots, x_d] \otimes RS_d).$$

Hence if we can find elements $z_1 \in \mathrm{F}_{i_1}\, Z(H_d^f), \ldots, z_m \in \mathrm{F}_{i_m}\, Z(H_d^f)$ with the property that $\operatorname{gr}_{i_1} z_1, \ldots, \operatorname{gr}_{i_m} z_m$ is a basis for $Z(R_l[x_1, \ldots, x_d] \otimes RS_d)$, then it follows immediately that $z_1, \ldots, z_m$ also is a basis for $Z(H_d^f)$. This is exactly what we are going to do. Recall the elements $y_i(k)$ and $m_d(\mu)$ of $R_l[x_1, \ldots, x_d] \otimes RS_d$ from the previous section.

Lemma 3.1. *Assume that $1 \leq i \leq d$ and $k = (a-1)l + r$ for some $a \geq 1$ and $0 \leq r < l$. Then we have that $x_i^k \in \mathrm{F}_{(a-1)(l-1)+r}\, H_d^f$ and*

$$\operatorname{gr}_{(a-1)(l-1)+r} x_i^k = \begin{cases} y_i(k) + (*) & \text{if } r = 0, \\ y_i(k) & \text{if } r > 0, \end{cases}$$

where $(*)$ denotes a linear combination of products of disjoint colored cycles of the form $A_1^{(l-1)} \cdots A_m^{(l-1)}$ such that $i \in A_1 \cup \cdots \cup A_m \subseteq \{1, \ldots, i\}$ and $|A_1 \cup \cdots \cup A_m| \leq a - 1$.

Proof. Assume to start with that $k = l$, i.e. $a = 2, r = 0$. We prove the lemma in this case by induction on $i = 1, \ldots, d$. For the base case, we have that $x_1^l = -c_1 x_1^{l-1} - \cdots - c_l$, so it is in filtered degree $(l-1)$ and $\operatorname{gr}_{l-1} x_1^l = -c_1(1)^{(l-1)} = y_1(l) - c_1(1)^{(l-1)}$. For the induction step, we have by the relations that

$$x_{i+1}^l = s_i x_i^l s_i + \sum_{t=0}^{l-1} x_i^t x_{i+1}^{l-1-t} s_i.$$

Hence by induction we get that x_{i+1}^l is in filtered degree $(l-1)$ and

$$\begin{aligned} \operatorname{gr}_{l-1} x_{i+1}^l &= (i\, i{+}1)(y_i(l) - c_1(i)^{(l-1)})(i\, i{+}1) + (i\, i{+}1)^{(0)} \\ &= y_{i+1}(l) - c_1(i+1)^{(l-1)} \end{aligned}$$

as we wanted.

Now assume that $k = (a-1)l$ for any $a \geq 1$, i.e. the case when $r = 0$. By the previous paragraph, we have that $x_i^k = (x_i^l)^{a-1}$ is in filtered degree $(a-1)(l-1)$ and

$$\operatorname{gr}_{(a-1)(l-1)} x_i^k = (y_i(l) - c_1(i)^{(l-1)})^{a-1}.$$

By Lemma 2.6 this equals $y_i(k) + (*)$ where $(*)$ is a linear combination of products of disjoint colored cycles of the form $A_1^{(l-1)} \cdots A_m^{(l-1)}$ such that $i \in A_1 \cup \cdots \cup A_m \subseteq \{1, \ldots, i\}$ and $|A_1 \cup \cdots \cup A_m| \leq a - 1$.

Finally assume that $k = (a-1)l + r$ for $0 < r < l$. Writing $x_i^k = (x_i^{(a-1)l})(x_i^r)$ and using the previous paragraph and Lemma 2.2 gives the desired conclusion in this case. $\square$

For any d-tuple $\mu = (\mu_1, \ldots, \mu_d)$ of non-negative integers, let

$$p_d(\mu) := \sum_{\nu \sim \mu} x_1^{\nu_1} \cdots x_d^{\nu_d} \in H_d^f.$$

Since this is a symmetric polynomial in $x_1, \ldots, x_d$, it is automatically central. Theorem 1 from the introduction is a consequence of the following more precise result.

Theorem 3.2. *For $\mu \in \mathcal{P}_d(l)$, we have that $p_d(\mu) \in \mathrm{F}_r\, Z(H_d^f)$ where $r = |\mu| - |\mu/l|$. Moreover, $\mathrm{gr}_r\, p_d(\mu) = m_d(\mu) + (*)$ where $(*)$ is a linear combination of $m_d(\nu)$'s for $\nu \in \mathcal{P}_d(l)$ with $|\nu/l| + \ell(\nu) < |\mu/l| + \ell(\mu)$. Hence, $\mathrm{gr}\, Z(H_d^f) = Z(R_l[x_1, \ldots, x_d] \otimes RS_d)$ and the elements*

$$\{p_d(\mu) \mid \mu \in \mathcal{P}_d(l)\}$$

form a basis for $Z(H_d^f)$. In particular, $Z(H_d^f)$ is a free R-module of rank equal to the number of l-multipartitions of d.

Proof. Recall the bijection $\varphi : \mathcal{M}_d(l) \to \mathcal{P}_d(l)$ and also the notation $\#\boldsymbol{\lambda}$ from the last paragraph of the proof of Theorem 2.7. We showed there for $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$ with $\varphi(\boldsymbol{\lambda}) = \mu$ that $m_d(\mu) = z_d(\boldsymbol{\lambda}) + (*)$ where $(*)$ is a linear combination of $z_d(\boldsymbol{\nu})$'s with $\#\boldsymbol{\nu} < \#\boldsymbol{\lambda}$. Note $\#\boldsymbol{\lambda} = |\mu/l| + \ell(\mu)$. So we get from this also that $z_d(\boldsymbol{\lambda}) = m_d(\mu) + (*)$ where $(*)$ is a linear combination of $m_d(\nu)$'s for $\nu \in \mathcal{P}_d(l)$ with $|\nu/l| + \ell(\nu) < |\mu/l| + \ell(\mu)$.

Now, by Lemma 3.1 and the definitions, $p_d(\mu)$ is in filtered degree $r = |\mu| - |\mu/l|$ and moreover $\mathrm{gr}_r\, p_d(\mu) = m_d(\mu) + (*)$ where $(*)$ is a linear combination of products of disjoint colored cycles $A_1^{(r_1)} \cdots A_m^{(r_m)}$ such that $|A_1 \cup \cdots \cup A_m| < |\mu/l| + \ell(\mu)$. Since $\mathrm{gr}_r\, p_d(\mu)$ is central, it follows by Lemma 2.2 and Theorem 2.4 that $(*)$ can be rewritten as a linear combination of $z_d(\boldsymbol{\nu})$'s with $\#\boldsymbol{\nu} < \#\boldsymbol{\lambda}$. Hence by the first paragraph it is also a linear combination of $m_d(\nu)$'s with $|\nu/l| + \ell(\nu) < |\mu/l| + \ell(\mu)$. This proves that the elements

$$\left\{\mathrm{gr}_{|\mu|-|\mu/l|}\, p_d(\mu) \mid \mu \in \mathcal{P}_d(l)\right\}$$

form a basis for $Z(R_l[x_1, \ldots, x_d] \otimes RS_d)$. Now the theorem follows by the general principles discussed just before Lemma 3.1. $\square$

Corollary 3.3. *If $d!$ is invertible in R, then the center of H_d^f is generated by the power sums $x_1^r + \cdots + x_d^r$ for $1 \le r \le d$.*

Proof. Under the assumption on R, it is well known that every symmetric polynomial in variables $x_1, \ldots, x_d$ can be expressed as a polynomial in the first r power sums. $\square$

4. The blocks of H_d^f

In this section, we replace the ground ring R with a ground field F such that we can factor $f(x) = (x - q_1) \cdots (x - q_l)$ for $q_1, \ldots, q_l \in F$. We will denote the algebra H_d^f instead by $H_d^{\boldsymbol{q}}$ where $\boldsymbol{q} = (q_1, \ldots, q_l) \in F^l$. We point out that F is a splitting field for the algebra $H_d^{\boldsymbol{q}}$; one way to see this is to check that the construction of the irreducible $H_d^{\boldsymbol{q}}$-modules over the algebraic closure of F from [K, §5.4] already makes sense over F itself. Theorem 1 just proved shows in particular that the dimension of $Z(H_d^{\boldsymbol{q}})$ is equal to the number of l-multipartitions of d. The goal in this section is to refine this statement by computing the dimensions of the centers of the individual blocks.

Before we can even formulate the result, we need an explicit combinatorial parametrization of the blocks, or equivalently, the central characters of $H_d^{\boldsymbol{q}}$. This is a well known consequence of Theorem 1. To start with we recall the classification of central characters of H_d itself following [K, §4.2]. Given a tuple $\boldsymbol{i} = (i_1, \ldots, i_d) \in F^d$, write

$$\chi(\boldsymbol{i}) : Z(H_d) \to F$$

for the central character mapping a symmetric polynomial $f(x_1, \ldots, x_d)$ to $f(i_1, \ldots, i_d)$. Clearly, $\chi(\boldsymbol{i}) = \chi(\boldsymbol{j})$ if and only if $\boldsymbol{i} \sim \boldsymbol{j}$, so this gives a parametrization of central characters of H_d by the set X_d of $\sim$-equivalence classes in F^d. Now we pass to the quotient $H_d^{\boldsymbol{q}}$ of H_d. Since $Z(H_d)$ maps surjectively onto $Z(H_d^{\boldsymbol{q}})$ by Theorem 1, the set of all central characters of $H_d^{\boldsymbol{q}}$ is naturally parametrized by the subset

$$X_d^{\boldsymbol{q}} = \{\boldsymbol{i} \in X_d \mid \chi(\boldsymbol{i}) : Z(H_d) \to F \text{ factors through the quotient } Z(H_d^{\boldsymbol{q}})\}$$

of X_d. To complete the classification of blocks of $H_d^{\boldsymbol{q}}$, it just remains to describe this subset $X_d^{\boldsymbol{q}}$ combinatorially.

To do this, we must first construct enough central characters, which we do by considering *dual Specht modules* following [BK, §6]. For a partition λ of d, let S^λ denote the usual Specht module for the symmetric group S_d over the field F and $S_\lambda = (S^\lambda)^*$ be its dual. Given any $q \in F$, we can extend S_λ to a module over the degenerate affine Hecke algebra H_d so that x_1 acts by scalar multiplication by q. We denote the resulting H_d-module by S_λ^q. If $d = d' + d''$, there is a natural embedding of $H_{d'} \otimes H_{d''}$ into H_d, so it makes sense to define the product

$$M' \circ M'' = H_d \otimes_{H_{d'} \otimes H_{d''}} (M' \boxtimes M'')$$

of an $H_{d'}$-module M' and an $H_{d''}$-module M'', where $\boxtimes$ denotes outer tensor product. Given an l-multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ of d, the H_d-module

$$S_{\boldsymbol{\lambda}}^{\boldsymbol{q}} := S_{\lambda^{(1)}}^{q_1} \circ \cdots \circ S_{\lambda^{(l)}}^{q_l}$$

factors through the quotient $H_d^{\boldsymbol{q}}$ to give a well-defined $H_d^{\boldsymbol{q}}$-module. This is the *dual Specht module* parametrized by the multipartition $\boldsymbol{\lambda}$.

Let us compute the central character of the dual Specht module $S_{\boldsymbol{q}}^{\boldsymbol{\lambda}}$. Note that if M' is an $H_{d'}$-module of central character $\chi(\boldsymbol{i}')$ and M'' is an $H_{d''}$-module of central character $\chi(\boldsymbol{i}'')$, then $M' \circ M''$ is of central character $\chi(\boldsymbol{i}' \circ \boldsymbol{i}'')$ where $\boldsymbol{i}' \circ \boldsymbol{i}''$ denotes the concatenation $(i'_1, \ldots, i'_{d'}, i''_1, \ldots, i'_{d''})$. This reduces to the problem of computing the central character simply of S_q^{λ} for a partition λ of d, which is well known: for each $i, j \geq 1$ fill the box in the ith row and jth column of the Young diagram of λ with the *residue* $(q + i - j)$; then S_{λ}^q is of central character parametrized by the tuple $\boldsymbol{i}_{\lambda}^q$ obtained by reading off the residues in all the boxes in some order. For example, if $\lambda = (4, 2, 1)$ and $q = 5$ then the residues are

5	6	7	8
4	5		
3			

so S_{λ}^q is of central character parametrized by $\boldsymbol{i}_{\lambda}^q \sim (5, 6, 7, 8, 4, 5, 3)$. Given $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)})$, we deduce that the central character of $S_{\boldsymbol{\lambda}}^{\boldsymbol{q}}$ is $\chi(\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}})$ where

$$\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = \boldsymbol{i}_{\lambda^{(1)}}^{q_1} \circ \cdots \circ \boldsymbol{i}_{\lambda^{(l)}}^{q_l} \,.$$

In this way, we have proved the existence of many central characters of $H_d^{\boldsymbol{q}}$.

Now we proceed like in finite group theory. Let $\bar{R}$ be a Noetherian domain with maximal ideal $\bar{\mathfrak{m}}$ such that $F = \bar{R}/\bar{\mathfrak{m}}$ and the field of fractions of $\bar{R}$ is of characteristic 0. Let $\hat{q}_1, \ldots, \hat{q}_l \in \bar{R}$ be lifts of the parameters $q_1, \ldots, q_l \in F$. Let R be the ring obtained by first localizing the polynomial algebra $\bar{R}[t_1, \ldots, t_l]$ at the maximal ideal generated by $\bar{\mathfrak{m}}$ and $t_1 - \hat{q}_1, \ldots, t_l - \hat{q}_l$, and then completing with respect to the image of this maximal ideal. We still have that $F = R/\mathfrak{m}$, where $\mathfrak{m}$ is the unique maximal ideal of R. Also let K be the field of fractions of R. Letting $\boldsymbol{t} = (t_1, \ldots, t_l)$, define $H_d^{\boldsymbol{t}}$ and $\mathcal{H}_d^{\boldsymbol{t}}$ to be the degenerate cyclotomic Hecke algebras defined by the polynomial $f(x) = (x - t_1) \cdots (x - t_l)$ over the field K and over the ring R, respectively. In view of the PBW theorem for degenerate cyclotomic Hecke algebras, $H_d^{\boldsymbol{t}}$ is naturally isomorphic to $K \otimes_R \mathcal{H}_d^{\boldsymbol{t}}$, and $H_d^{\boldsymbol{q}}$ is naturally isomorphic to $F \otimes_R \mathcal{H}_d^{\boldsymbol{t}}$, viewing F here as an R-module so that each t_i acts as multiplication by q_i. Note moreover that the definition of dual Specht modules carries over unchanged to give modules $S_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$ for $H_d^{\boldsymbol{t}}$ and $\mathcal{S}_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$ for $\mathcal{H}_d^{\boldsymbol{t}}$ for each $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$, such that $S_{\boldsymbol{\lambda}}^{\boldsymbol{t}} \cong K \otimes_R \mathcal{S}_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$ and $S_{\boldsymbol{\lambda}}^{\boldsymbol{q}} \cong F \otimes_R \mathcal{S}_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$. The following lemma is well known, but the proof given here is quite instructive.

Lemma 4.1. *The algebra $H_d^{\boldsymbol{t}}$ is split semisimple. Moreover, the dual Specht modules $S_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$ give a complete set of pairwise non-isomorphic irreducible $H_d^{\boldsymbol{t}}$-modules.*

Proof. For each $i = 1, \ldots, l$, let $H_d^{t_i}$ denote the degenerate cyclotomic Hecke algebra over K defined by the polynomial $f(x) = (x - t_i)$. There is an isomorphism $H_d^{t_i} \xrightarrow{\sim} KS_d$ which is the identity on S_d and maps x_1 to t_i. Since K is a field of charcteristic zero, we get from this that each $H_d^{t_i}$ is a

split semisimple algebra and, by the classical representation theory of the symmetric group, the dual Specht modules $S_\lambda^{t_i}$ for all partitions λ of d give a complete set of pairwise non-isomorphic irreducible $H_d^{t_i}$-modules. Since $t_1, \ldots, t_l$ are algebraically independent, the proof of [BK, Corollary 5.20] shows that there is an isomorphism

$$H_d^{\boldsymbol{t}} \cong \bigoplus_{d_1 + \cdots + d_l = d} H_{d_1}^{t_1} \otimes \cdots \otimes H_{d_l}^{t_l}$$

under which $S_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$ corresponds to the outer tensor product $S_{\lambda^{(1)}}^{t_1} \boxtimes \cdots \boxtimes S_{\lambda^{(l)}}^{t_l}$ of dual Specht modules. The lemma follows. $\quad\square$

Lemma 4.1 implies that all the dual Specht modules $\{S_{\boldsymbol{\lambda}}^{\boldsymbol{t}} \mid \boldsymbol{\lambda} \in X_d^{\boldsymbol{t}}\}$ have different central characters. One can also see this directly by observing from the combinatorial definition that the tuples $\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$ for $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$ are in different $\sim$-equivalence classes, i.e. in the generic case the map

$$\mathcal{M}_d(l) \to X_d^{\boldsymbol{t}}, \qquad \boldsymbol{\lambda} \mapsto \boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$$

is injective. Actually, it is a bijection, by a trivial special case of the following lemma completing the classification of blocks of $H_d^{\boldsymbol{q}}$ in general.

Lemma 4.2. $X_d^{\boldsymbol{q}} = \{\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} \mid \boldsymbol{\lambda} \in \mathcal{M}_d(l)\}$.

Proof. We have already noted that all $\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}}$ belong to $X_d^{\boldsymbol{q}}$. Conversely, we need show that the $\chi(\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}})$'s give all of the central characters of $H_d^{\boldsymbol{q}}$. This follows from the following claim: we have that $\prod_{\boldsymbol{\lambda} \in \mathcal{M}_d(l)}(z - \chi(\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}})(z)) = 0$ for every $z \in Z(H_d^{\boldsymbol{q}})$. Note the following diagram commutes

$$
\begin{array}{ccc}
Z(\mathcal{H}_d^{\boldsymbol{t}}) & \xrightarrow{\chi(\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{t}})} & R \\
\downarrow & & \downarrow \\
Z(H_d^{\boldsymbol{q}}) & \xrightarrow[\chi(\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}})]{} & F
\end{array}
$$

where the vertical maps are defined by evaluating each t_i at q_i. So the claim follows if we can show that $\prod_{\boldsymbol{\lambda} \in \mathcal{M}_d(l)}(z - \chi(\boldsymbol{i}_{\boldsymbol{\lambda}})(z)) = 0$ for every $z \in \mathcal{H}_d^{\boldsymbol{t}}$. But we have that $\mathcal{H}_d^{\boldsymbol{t}} \subseteq H_d^{\boldsymbol{t}}$, and in the semisimple algebra $H_d^{\boldsymbol{t}}$ it is certainly the case that $\prod_{\boldsymbol{\lambda} \in \mathcal{M}_d(l)}(z - \chi(\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{t}})(z)) = 0$ because the $\chi(\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{t}})$'s for all $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$ are the central characters of a full set of irreducible $H_d^{\boldsymbol{t}}$-modules, thanks to Lemma 4.1. $\quad\square$

For $\boldsymbol{i} \in X_d^{\boldsymbol{q}}$, let $b(\boldsymbol{i})$ be the primitive central idempotent corresponding to the central character $\chi(\boldsymbol{i})$, that is, $b(\boldsymbol{i})$ is the unique element of $Z(H_d^{\boldsymbol{q}})$ that acts as one on irreducible modules of central character $\chi(\boldsymbol{i})$ and as zero on all other irreducibles. Thus, we have that

$$H_d^{\boldsymbol{q}} = \bigoplus_{\boldsymbol{i} \in X_d^{\boldsymbol{q}}} b(\boldsymbol{i}) H_d^{\boldsymbol{q}}.$$

This is the decomposition of $H_d^{\boldsymbol{q}}$ into blocks. Similarly, recalling the bijection $\mathcal{M}_d(l) \to X_d^{\boldsymbol{t}}, \boldsymbol{\lambda} \mapsto \boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$, we can define idempotents $b(\boldsymbol{\lambda}) \in Z(H_d^{\boldsymbol{t}})$ for each $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$ such that $b(\boldsymbol{\lambda})$ acts as one on $S_{\boldsymbol{\lambda}}^{\boldsymbol{t}}$ and as zero on all other dual Specht modules. Of course, the resulting decomposition

$$H_d^{\boldsymbol{t}} = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{M}_d(l)} b(\boldsymbol{\lambda}) H_d^{\boldsymbol{t}}$$

is the Wedderburn decomposition of the semisimple algebra $H_d^{\boldsymbol{t}}$.

Since R is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}$, and moreover we know that $Z(\mathcal{H}_d^{\boldsymbol{t}})$ surjects onto $Z(H_d^{\boldsymbol{q}})$ by Theorem 1, there is a unique lift of each $b(\boldsymbol{i}) \in Z(H_d^{\boldsymbol{q}})$ to a central idempotent $\hat{b}(\boldsymbol{i}) \in \mathcal{H}_d^{\boldsymbol{t}}$; see e.g. [E, Corollary 7.5]. This lifts the block decomposition of $H_d^{\boldsymbol{q}}$ to a decomposition

$$H_d^{\boldsymbol{t}} = \bigoplus_{\boldsymbol{i} \in X_d^{\boldsymbol{q}}} \hat{b}(\boldsymbol{i}) H_d^{\boldsymbol{t}}$$

of the semisimple algebra $H_d^{\boldsymbol{t}}$. Finally, the commutative diagram from the proof of Lemma 4.2 implies for each $\boldsymbol{i} \in X_d^{\boldsymbol{q}}$ that

$$\hat{b}(\boldsymbol{i}) H_d^{\boldsymbol{t}} = \bigoplus_{\substack{\boldsymbol{\lambda} \in \mathcal{M}_d(l) \\ \boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = \boldsymbol{i}}} b(\boldsymbol{\lambda}) H_d^{\boldsymbol{t}}.$$

Now we can prove the only new result of the section, as follows.

Theorem 4.3. *For $\boldsymbol{i} \in X_d^{\boldsymbol{q}}$, the dimension of the center of the block $b(\boldsymbol{i}) H_d^{\boldsymbol{q}}$ is equal to the number of l-multipartitions $\boldsymbol{\lambda}$ of d such that $\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = \boldsymbol{i}$.*

Proof. By Theorem 1, $Z(\mathcal{H}_d^{\boldsymbol{t}})$ is a free R-module of finite rank. Since R is a local ring, it follows that the summand $Z(\hat{b}(\boldsymbol{i})\mathcal{H}_d^{\boldsymbol{t}}) = \hat{b}(\boldsymbol{i})Z(\mathcal{H}_d^{\boldsymbol{t}})$ is also free, of rank equal to $\dim_K Z(\hat{b}(\boldsymbol{i})H_d^{\boldsymbol{t}})$. Since each $b(\boldsymbol{\lambda})H_d^{\boldsymbol{t}}$ is a full matrix algebra with a one dimensional center, we know from the preceeding discussion that this dimension is equal to the number of l-multipartitions $\boldsymbol{\lambda}$ with $\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = \boldsymbol{i}$. By Theorem 1 again, the isomorphism $F \otimes_R \mathcal{H}_d^{\boldsymbol{t}} \xrightarrow{\sim} H_d^{\boldsymbol{q}}$ induces an isomorphism $F \otimes_R Z(\mathcal{H}_d^{\boldsymbol{t}}) \xrightarrow{\sim} Z(H_d^{\boldsymbol{q}})$. From this, we get an isomorphism $F \otimes_R Z(\hat{b}(\boldsymbol{i})\mathcal{H}_d^{\boldsymbol{t}}) \xrightarrow{\sim} Z(b(\boldsymbol{i})H_d^{\boldsymbol{q}})$. So $\dim_F Z(b(\boldsymbol{i})H_d^{\boldsymbol{q}})$ is the same as the rank of $Z(\hat{b}(\boldsymbol{i})\mathcal{H}_d^{\boldsymbol{t}})$, i.e. the number of l-multipartitions $\boldsymbol{\lambda}$ of d with $\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = \boldsymbol{i}$. $\square$

5. The center of parabolic category $\mathcal{O}$

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ with natural module V. We denote the standard basis for V by $v_1, \ldots, v_n$ and use the notation $e_{i,j}$ for the matrix units in $\mathfrak{g}$. Let $\mathfrak{d}$ be the subalgebra of $\mathfrak{g}$ of diagonal matrices and $\mathfrak{b}$ be the standard Borel subalgebra of upper triangular matrices. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the basis for $\mathfrak{d}^*$ dual to the standard basis $e_{1,1}, \ldots, e_{n,n}$ for $\mathfrak{d}$. We write $L(\alpha)$ for the irreducible highest weight module of highest weight $(\alpha - \rho)$, where ρ is the

weight $-\varepsilon_2 - 2\varepsilon_3 - \cdots - (n-1)\varepsilon_n$. Viewing elements of $S(\mathfrak{d})$ as polynomial functions on $\mathfrak{d}^*$, the *Harish-Chandra homomorphism*

$$\Psi : Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{d})^{S_n}$$

can be defined by declaring that $\Psi(z)$ is the unique element of $S(\mathfrak{d})$ with the property that z acts on $L(\alpha)$ by the scalar $(\Psi(z))(\alpha)$ for each $\alpha \in \mathfrak{d}^*$. Its image is the algebra $S(\mathfrak{d})^{S_n}$ of symmetric polynomials (for the usual action of S_n on $\mathfrak{d}$ not the dot action).

Letting $\tilde{e}_{i,j} := e_{i,j} + \delta_{i,j}(u+1-i)$ for short, it is classical that the coefficients $z_1, \ldots, z_n$ of the polynomial

$$z(u) = \sum_{r=0}^{n} z_r u^{n-r} := \sum_{w \in S_n} \operatorname{sgn}(w) \tilde{e}_{w1,1} \cdots \tilde{e}_{wn,n} \in U(\mathfrak{g})[u]$$

are algebraically independent generators for the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. We adopt the convention that $z_r = 0$ for $r > n$. The image of $z(u)$ under Ψ is given by the formula

$$\Psi(z(u)) = (u + e_{1,1}) \cdots (u + e_{n,n}).$$

Hence, for $\alpha = \sum_{i=1}^{n} a_i \varepsilon_i \in \mathfrak{d}^*$, the central element z_r acts on $L(\alpha)$ as the scalar $e_r(\alpha) = e_r(a_1, \ldots, a_n)$, the rth *elementary symmetric function* evaluated at the numbers $a_1, \ldots, a_n$. Let P denote the free abelian group on basis $\{\gamma_a \mid a \in \mathbb{C}\}$. For $\alpha = \sum_{i=1}^{n} a_i \varepsilon_i \in \mathfrak{d}^*$, let $\nu(\alpha) = \gamma_{a_1} + \cdots + \gamma_{a_n} \in P$. The point of this definition is that $L(\alpha)$ and $L(\beta)$ have the same central character if and only if $\nu(\alpha) = \nu(\beta)$. In this way, we have parametrized the central characters of $U(\mathfrak{g})$ by the set of all $\nu \in P$ whose coefficients are non-negative integers summing to d.

Let $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be the canonical comultiplication on the universal enveloping algebra of $\mathfrak{g}$. We are only going to need to work with the homomorphism $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \operatorname{End}_{\mathbb{C}}(V)$ obtained by composing Δ with the map $1 \otimes \varphi$ where $\varphi : U(\mathfrak{g}) \to \operatorname{End}_{\mathbb{C}}(V)$ here is the algebra homomorphism arising from the representation of $\mathfrak{g}$ on V. Also, let

$$\Omega = \sum_{i,j=1}^{n} e_{i,j} \otimes e_{j,i} \ \in U(\mathfrak{g}) \otimes \operatorname{End}_{\mathbb{C}}(V).$$

The following lemma is probably classical.

Lemma 5.1. *For $r \geq 0$, we have that*

$$\delta(z_r) = z_r \otimes 1 + \sum_{s=0}^{r-1} (-1)^s (z_{r-1-s} \otimes 1)\Omega^s.$$

Proof. Both sides of the equation are elements of $U(\mathfrak{g}) \otimes \operatorname{End}_{\mathbb{C}}(V)$, so can be viewed as $n \times n$ matrices with entries in $U(\mathfrak{g})$. To see that these matrices are equal, it suffices to check that their entries act in the same way on sufficiently many finite dimensional irreducible representations of $\mathfrak{g}$. This

reduces to the following problem. Take $\alpha = \sum_{i=1}^{n} a_i \varepsilon_i \in \mathfrak{d}^*$ such that $L(\alpha)$ is finite dimensional and

$$L(\alpha) \otimes V \cong \bigoplus_{i=1}^{n} L(\alpha + \varepsilon_i).$$

We need to show that the left and right hand sides of the given equation define the same endomorphism of $L(\alpha) \otimes V$. For such an α, let $M := L(\alpha) \otimes V$ and write v_+ for a highest weight vector in $L(\alpha)$. For $i = 0, \ldots, n$, define M_i to be the submodule of M generated by the vectors $v_+ \otimes v_j$ ($j \leq i$). Since $v_+ \otimes v_i$ is a highest weight vector of weight $\alpha + \varepsilon_i$ modulo M_{i-1}, the assumption on α implies that $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is a filtration of M such that $M_i/M_{i-1} \cong L(\alpha + \varepsilon_i)$. Since the filtration splits, there is a unique highest weight vector $x_i \in M$ such that $x_i \equiv v_+ \otimes v_i \pmod{M_{i-1}}$. Now we just check that the left and right hand sides of the given equation act on these highest weight vectors by the same scalar for each $i = 1, \ldots, n$. Of course $\delta(z_r)$ acts on x_i as $e_r(\alpha + \varepsilon_i)$, while each $z_t \otimes 1$ acts as $e_t(\alpha)$ on all of M. Finally, since Ω defines a $\mathfrak{g}$-module endomorphism of M, it leaves M_{i-1} invariant and maps x_i to a scalar multiple of itself. To compute the scalar, note that

$$\Omega(v_+ \otimes v_i) = \sum_{j \leq i} (e_{i,j} v_+) \otimes v_j = (e_{i,i} v_+) \otimes v_i + \sum_{j < i} (e_{i,j} v_+) \otimes v_j$$

$$= (a_i + i - 1) v_+ \otimes v_i + \sum_{j < i} (e_{i,j}(v_+ \otimes v_j) - v_+ \otimes v_i)$$

$$\equiv a_i v_+ \otimes v_i \pmod{M_{i-1}}.$$

Hence, $\Omega x_i = a_i x_i$. So the equation we are trying to prove reduces to checking that

$$e_r(\alpha + \varepsilon_i) = e_r(\alpha) + \sum_{s=0}^{r-1} (-1)^s e_{r-1-s}(\alpha) a_i^s$$

for each $i = 1, \ldots, n$. This follows from the following general identity which is true for all $r, k \geq 0$:

$$e_r(u_1, \ldots, u_k, u+1) = e_r(u_1, \ldots, u_k, u) + \sum_{s=0}^{r-1} (-1)^s e_{r-1-s}(u_1, \ldots, u_k, u) u^s.$$

To see this, expand both sides using the obvious formula $e_t(u_1, \ldots, u_k, v) = e_t(u_1, \ldots, u_k) + e_{t-1}(u_1, \ldots, u_k) v$. $\square$

Let M be any $\mathfrak{g}$-module. Recall from [AS, §2.2] that the degenerate affine Hecke algebra H_d over the ground field $\mathbb{C}$ acts naturally on the right on $M \otimes V^{\otimes d}$ by $\mathfrak{g}$-module endomorphisms. The action of each $w \in S_d$ arises from its usual action on $V^{\otimes d}$ by place permutation. The action of x_1 (from which one can deduce the action of all other x_i's) is the same as left multiplication by $\Omega \otimes 1^{\otimes(d-1)}$. For any partition μ with $\ell(\mu) \leq d$, recall the

notation $p_d(\mu)$ introduced just before Theorem 3.2; we are now viewing this expression as an element of H_d.

Lemma 5.2. *For any $r, d \geq 0$ and any highest weight module M of highest weight $\alpha - \rho \in \mathfrak{d}^*$, the endomorphism of $M \otimes V^{\otimes d}$ define by left multiplication by $z_r \in Z(\mathfrak{g})$ is equal to the endomorphism defined by right multiplication by*

$$\sum_\mu (-1)^{|\bar\mu|} \binom{d - \ell(\bar\mu)}{d - \ell(\mu)} e_{r-|\mu|}(\alpha) p_d(\bar\mu) \in Z(H_d)$$

where the sum is over partitions μ of length $\ell(\mu) \leq d$ and size $|\mu| \leq r$, and $\bar\mu$ denotes the partition $(\mu_1 - 1 \geq \cdots \geq \mu_{\ell(\mu)} - 1)$ obtained from μ by removing the first column of its diagram.

Proof. Let $\delta_d : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathrm{End}_\mathbb{C}(V)^{\otimes d}$ be the map defined inductively by setting $\delta_0 = 1$ and $\delta_d = (\delta \otimes 1^{\otimes(d-1)}) \circ \delta_{d-1}$ for $d \geq 1$. Let $\Omega_i := (\delta_{i-1} \otimes 1)(\Omega) \otimes 1^{(d-i)} \in U(\mathfrak{g}) \otimes \mathrm{End}_\mathbb{C}(V)^{\otimes d}$. If we adopt the convention that $(-\Omega)^{-1} = 1$, we can write the conclusion of Lemma 5.1 simply as

$$\delta(z_r) = \sum_{s=0}^r (z_{r-s} \otimes 1)(-\Omega)^{s-1}.$$

Proceeding from this by induction on d, it is straightforward to deduce that

$$\delta_d(z_r) = \sum_{\substack{s_1,\ldots,s_d \geq 0 \\ s_1 + \cdots + s_d \leq r}} (z_{r-s_1-\cdots-s_d} \otimes 1^{\otimes d})(-\Omega_1)^{s_1-1} \cdots (-\Omega_d)^{s_d-1},$$

interpreting the right hand side with same convention. Since $x_{i+1} = s_i x_i s_i + s_i$ and x_1 acts as Ω_1 by definition, one checks by induction that x_i acts as Ω_i for each i. Hence on applying our expression to $M \otimes V^{\otimes d}$, we deduce that z_r acts in the same way as

$$\sum_{\substack{s_1,\ldots,s_d \geq 0 \\ s_1 + \cdots + s_d \leq r}} e_{r-s_1-\cdots-s_d}(\alpha)(-x_1)^{s_1-1} \cdots (-x_d)^{s_d-1},$$

again interpreting $(-x_i)^{-1}$ as 1. It is now a combinatorial exercise to rewrite this expression as formulated in the statement of the lemma. $\square$

Corollary 5.3. *For any highest weight module M, the subalgebra of the algebra $\mathrm{End}_\mathbb{C}(M \otimes V^{\otimes d})^{\mathrm{op}}$ generated by the endomorphisms z_r $(1 \leq r \leq n)$ coincides with the subalgebra generated by $x_1^r + \cdots + x_d^r$ $(1 \leq r \leq d)$.*

Proof. Since we are working over a field of characteristic 0, any symmetric polynomial in $x_1, \ldots, x_d$ lies in the subalgebra generated by the power sums $x_1^r + \cdots + x_d^r$ $(1 \leq r \leq d)$. By Lemma 5.2, the endomorphism defined by z_r can be expressed as a symmetric polynomial in $x_1, \ldots, x_d$, so it lies in the subalgebra generated by the power sums.

Conversely, we show by induction on $r \geq 0$ that every homogeneous symmetric polynomial in $x_1, \ldots, x_d$ of degree r acts on $M \otimes V^{\otimes d}$ in the same

way as some element of the subalgebra generated by $z_1, \ldots, z_{r+1}$. For the induction step, every homogeneous symmetric polynomial of degree r lies in the subalgebra generated by the power sums $x_1^s + \cdots + x_d^s$ $(1 \leq s \leq r)$. By induction all of these power sums with $s < r$ certainly lie in the subalgebra generated by $z_1, \ldots, z_{r+1}$, so it just remains to show that $x_1^r + \cdots + x_d^r$ does too. By Lemma 5.2, the image of z_{r+1} is the same as the image of $x_1^r + \cdots + x_d^r$ (which is the term $p_d(\bar{\mu})$ when $\mu = (r+1)$) plus a linear combination of symmetric polynomials in $x_1, \ldots, x_d$ of strictly smaller degree, which we already have by the induction hypothesis. $\square$

Let $\mu = (\mu_1, \ldots, \mu_l)$ be a composition of n and let $\mathfrak{p}$ be the corresponding standard parabolic subalgebra of $\mathfrak{g}$ with standard Levi subalgebra $\mathfrak{h} \cong \mathfrak{gl}_{\mu_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{\mu_l}(\mathbb{C})$, as in the introduction. We are interested in the category $\mathcal{O}^\mu$ of all finitely generated $\mathfrak{g}$-modules that are locally finite over $\mathfrak{p}$ and integrable over $\mathfrak{h}$. Also let $\boldsymbol{q} = (\mu_1, \ldots, \mu_l)$ and let $H_d^{\boldsymbol{q}}$ denote the degenerate cyclotomic Hecke algebra from the previous section over the ground field $F = \mathbb{C}$. We are going to apply the Schur-Weyl duality for higher levels from [BK] (taking the choice of origin there to be $\boldsymbol{c} = (n, \ldots, n)$) to connect the category $\mathcal{O}^\mu$ to the finite dimensional algebras $H_d^{\boldsymbol{q}}$ for all $d \geq 0$. Actually, [BK] only considered the special case that μ is a partition, i.e. $\mu_1 \geq \cdots \geq \mu_l$, so we will need to extend some of the results of [BK] to the general case as we go.

To start with, we need some combinatorial definitions. Let

$$\mathrm{Col}^\mu = \left\{ \alpha = \sum_{i=1}^n a_i \varepsilon_i \in \mathfrak{d}^* \; \middle| \; \begin{array}{l} a_1, \ldots, a_n \in \mathbb{Z} \text{ such that } a_i > a_{i+1} \text{ for} \\ \text{all } i \neq \mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_l \end{array} \right\},$$

so called because its elements can be visualized as column strict tableaux of column shape μ like in [B1, §2]. The irreducible modules in $\mathcal{O}^\mu$ are the modules $\{L(\alpha) \mid \alpha \in \mathrm{Col}^\mu\}$. Hence the set $Y^\mu = \{\nu(\alpha) \mid \alpha \in \mathrm{Col}^\mu\}$ naturally parametrizes the central characters arising from modules in $\mathcal{O}^\mu$. Given $\nu \in Y^\mu$, we let $\mathrm{Col}_\nu^\mu = \{\alpha \in \mathrm{Col}^\mu \mid \nu(\alpha) = \nu\}$ and define $\mathcal{O}_\nu^\mu$ to be the Serre subcategory of $\mathcal{O}^\mu$ generated by the modules $\{L(\alpha) \mid \alpha \in \mathrm{Col}_\nu^\mu\}$. The category $\mathcal{O}^\mu$ then decomposes as

$$\mathcal{O}^\mu = \bigoplus_{\nu \in Y^\mu} \mathcal{O}_\nu^\mu.$$

This is the same as the central character decomposition of $\mathcal{O}^\mu$ that was described in the introduction.

Let $\gamma \in Y^\mu$ be the special element $\gamma = \sum_{i=1}^l \sum_{a=1}^{\mu_i} \gamma_a$. The key feature of γ is that the set Col_γ^μ contains just one weight α. In other words, for this α, $L(\alpha)$ is the unique irreducible module in $\mathcal{O}^\mu$ with central character parametrized by γ. This special irreducible module, which we denote

henceforth by P^μ, is automatically projective. For $d \geq 0$, let

$$Y_d^\mu = \{\gamma - (\gamma_{i_1} - \gamma_{i_1+1}) - \cdots - (\gamma_{i_d} - \gamma_{i_d+1}) \in Y^\mu \mid i_1, \ldots, i_d \in \mathbb{Z}, \},$$
$$\mathrm{Col}_d^\mu = \{\alpha \in \mathrm{Col}^\mu \mid \nu(\alpha) \in Y_d^\mu\}.$$

The irreducible modules $\{L(\alpha) \mid \alpha \in \mathrm{Col}_d^\mu\}$ are significant because they are exactly the irreducible constituents of the module $P^\mu \otimes V^{\otimes d}$. This statement is proved in [BK, §4] in the case that μ is a partition, and the same argument works in general.

Lemma 5.4. *The map $Y_d^\mu \to X_d^{\boldsymbol{q}}$ sending $\nu = \gamma - (\gamma_{i_1} - \gamma_{i_1+1}) - \cdots - (\gamma_{i_d} - \gamma_{i_d+1})$ to $\boldsymbol{i} = (i_1, \ldots, i_d)$ is injective with image equal to*

$$\{\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} \mid \boldsymbol{\lambda} \in \mathcal{M}_d(l) \text{ such that } \ell(\lambda^{(r)}) \leq \mu_r \text{ for } r = 1, \ldots, l\}.$$

For $\nu \in Y_d^\mu$ corresponding to $\boldsymbol{i} \in X_d^{\boldsymbol{q}}$ in this way, the map $\mathrm{Col}_\nu^\mu \to \mathcal{M}_d(l)$ sending $\alpha = \sum_{i=1}^n a_i \varepsilon_i$ to the multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ such that $\lambda^{(r)} = (a_{\mu_1+\cdots+\mu_{r-1}+1} - \mu_r, a_{\mu_1+\cdots+\mu_{r-1}+2} - \mu_{r-1}, \ldots, a_{\mu_1+\cdots+\mu_r} - 1)$ is injective with image equal to $\{\boldsymbol{\lambda} \in \mathcal{M}_d(l) \mid \boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = \boldsymbol{i}\}$.

Proof. We leave this as a simply combinatorial exercise. It is helpful to use the interpretation of Col_ν^μ as the set of column strict tableaux of column shape μ and type ν as in [BK, §4]. $\square$

As explained before Lemma 5.2, the degenerate affine Hecke algebra H_d acts on the right on $P^\mu \otimes V^{\otimes d}$ by $\mathfrak{g}$-module endomorphisms. We let

$$\rho^\mu : H_d \to \mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$$

be the resulting algebra homomorphism. The following lemma is the key to extending the results from [BK] to general μ.

Lemma 5.5. *Given another composition $\mu' \sim \mu$, there is an algebra isomorphism $\iota_{\mu,\mu'}$ making the following diagram commute:*

$$
\begin{array}{ccc}
& H_d & \\
{\scriptstyle \rho^\mu} \swarrow & & \searrow {\scriptstyle \rho^{\mu'}} \\
\mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}} & \xrightarrow[\iota_{\mu,\mu'}]{\sim} & \mathrm{End}_\mathfrak{g}(P^{\mu'} \otimes V^{\otimes d})^{\mathrm{op}}.
\end{array}
$$

Moreover, $\iota_{\mu,\mu'}$ intertwines the natural actions of $Z(\mathfrak{g})$ on the two endomorphism algebras.

Proof. We appeal to an argument due to Mazorchuk and Stroppel. By the proof of [MS, Theorem 5.4], there is an adjoint pair (F, G) of functors between the bounded derived categories

$$D^b(\mathcal{O}^\mu) \underset{G}{\overset{F}{\rightleftarrows}} D^b(\mathcal{O}^{\mu'}).$$

with the following properties:

(i) F and G commute with tensoring with finite dimensional $\mathfrak{g}$-modules, that is, for any finite dimensional $\mathfrak{g}$-module V there are given natural isomorphisms $\alpha_V : F \circ ? \otimes V \to ? \otimes V \circ F$ and $\beta_V : G \circ ? \otimes V \to ? \otimes V \circ G$ such that the following two diagrams commute for any morphism $f : V \to W$ of finite dimensional $\mathfrak{g}$-modules:

$$
\begin{array}{ccc}
F \circ ? \otimes V & \xrightarrow{\ \alpha_V\ } & ? \otimes V \circ F \\
{\scriptstyle 1(\mathrm{id}\otimes f)}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id}\otimes f)1} \\
F \circ ? \otimes W & \xrightarrow[\ \alpha_W\ ]{} & ? \otimes W \circ F
\end{array}
\qquad
\begin{array}{ccc}
G \circ ? \otimes V & \xrightarrow{\ \beta_V\ } & ? \otimes V \circ G \\
{\scriptstyle 1(\mathrm{id}\otimes f)}\big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id}\otimes f)1} \\
G \circ ? \otimes W & \xrightarrow[\ \beta_W\ ]{} & ? \otimes W \circ G
\end{array}
$$

(ii) The isomorphisms α_V and β_V are compatible with the unit $\eta : \mathrm{Id} \to G \circ F$ and counit $\varepsilon : F \circ G \to \mathrm{Id}$ of the canonical adjunction between F and G, i.e. the following diagrams commute:

$$
\begin{array}{ccc}
? \otimes V & \xrightarrow{\ 1\eta\ } & ? \otimes V \circ G \circ F \\
{\scriptstyle \eta 1}\big\downarrow & & \big\uparrow{\scriptstyle \beta_V 1} \\
G \circ F \circ ? \otimes V & \xrightarrow[\ 1\alpha_V\ ]{} & G \circ ? \otimes V \circ F
\end{array}
\qquad
\begin{array}{ccc}
? \otimes V & \xleftarrow{\ 1\varepsilon\ } & ? \otimes V \circ F \circ G \\
{\scriptstyle \varepsilon 1}\big\uparrow & & \big\uparrow{\scriptstyle \alpha_V 1} \\
F \circ G \circ ? \otimes V & \xrightarrow[\ 1\beta_V\ ]{} & F \circ ? \otimes V \circ G
\end{array}
$$

(iii) F and G restrict to mutually inverse equivalences of categories between $\mathcal{O}_\gamma^\mu$ and $\mathcal{O}_\gamma^{\mu'}$.

(iv) The following associativity pentagon commutes for any two finite dimensional $\mathfrak{g}$-modules V and W:

$$
\begin{array}{ccc}
F \circ ? \otimes (V \otimes W) & \xrightarrow{\quad\quad\quad \alpha_{V\otimes W} \quad\quad\quad} & ? \otimes (V \otimes W) \circ F \\
\big\| & & \big\| \\
F \circ ? \otimes W \circ ? \otimes V \xrightarrow[\ \alpha_W 1\ ]{} ? \otimes W \circ F \circ ? \otimes V & \xrightarrow[\ 1\alpha_V\ ]{} & ? \otimes W \circ ? \otimes V \circ F
\end{array}
$$

(v) F transforms the endomorphism of an object M defined by left multiplication by $z \in Z(\mathfrak{g})$ to the endomorphism of FM defined by left multiplication by the same element z.

By (iii), we can choose an isomorphism $F(P^\mu) \cong P^{\mu'}$ allowing us to simply identify $F(P^\mu)$ with $P^{\mu'}$. The isomorphism $\alpha_{V^{\otimes d}}$ then allows us to identify $F(P^\mu \otimes V^{\otimes d})$ with $P^{\mu'} \otimes V^{\otimes d}$. Applying G to this, we get an isomorphism $G(F(P^\mu \otimes V^{\otimes d}) \cong G(P^{\mu'} \otimes V^{\otimes d})$, hence on composing with the counit of the adjunction we get a map $P^\mu \otimes V^{\otimes d} \to G(P^{\mu'} \otimes V^{\otimes d})$ which by (ii) and (iii) is an isomorphism. Using this, we also identify $G(P^{\mu'} \otimes V^{\otimes d})$ with $P^\mu \otimes V^{\otimes d}$. Now the functor F defines an algebra homomorphism

$$
\iota_{\mu,\mu'} : \mathrm{End}_{\mathfrak{g}}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}} \to \mathrm{End}_{\mathfrak{g}}(P^{\mu'} \otimes V^{\otimes d})^{\mathrm{op}}
$$

and the functor G defines a homomorphism

$$
\iota_{\mu',\mu} : \mathrm{End}_{\mathfrak{g}}(P^{\mu'} \otimes V^{\otimes d})^{\mathrm{op}} \to \mathrm{End}_{\mathfrak{g}}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}
$$

such that $\iota_{\mu,\mu'}$ and $\iota_{\mu',\mu}$ are mutual inverses. Moreover, by (v), both homomorphisms intertwine the natural actions of $Z(\mathfrak{g})$.

It just remains to check that $\iota_{\mu,\mu'}$ is compatible with the action of H_d. The compatibility of $\iota_{\mu,\mu'}$ with the action of each $w \in S_d$ follows immediately from the naturality in (i). So it suffices to show that it is compatible with the action of x_1. For this, we first reduce using (iv) to checking compatiblity just in the special case $d = 1$. In that case it follows from (v) since by Lemma 5.1 we have that x_1 acts as left multiplication by $\Omega = z_2 \otimes 1 + z_1 \otimes 1 - \delta(z_2)$, and $z_2 \otimes 1$ and $z_1 \otimes 1$ act by the same scalars on $P^\mu \otimes V$ and $P^{\mu'} \otimes V$. $\quad\square$

We can formulate the critical result needed from [BK] as follows.

Theorem 5.6. *The image of $\rho^\mu : H_d \to \mathrm{End}_\mathbb{C}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$ coincides with the endomorphism algebra $\mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$. Moreover, the representation ρ^μ factors through the quotient $H_d^{\boldsymbol{q}}$ of H_d, and the kernel of the induced map $H_d^{\boldsymbol{q}} \twoheadrightarrow \mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$ is generated by $(1-e)$, where $e \in H_d^{\boldsymbol{q}}$ is the central idempotent $e = \sum_{\boldsymbol{i}} b(\boldsymbol{i})$ summing over all $\boldsymbol{i}$ lying in the set*

$$\{\boldsymbol{i}_\lambda^{\boldsymbol{q}} \mid \boldsymbol{\lambda} \in \mathcal{M}_d(l) \text{ such that } \ell(\lambda^{(r)}) \leq \mu_r \text{ for } r = 1,\dots,l\}$$

from Lemma 5.4. Hence, ρ^μ induces an isomorphism between the sum of blocks $eH_d^{\boldsymbol{q}}$ of $H_d^{\boldsymbol{q}}$ and the endomorphism algebra $\mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$.

Proof. If μ is a partition, this follows by [BK, Theorem 5.13] and [BK, Corollary 6.7]. It then follows for arbitrary μ too by Lemma 5.5. $\quad\square$

Corollary 5.7. *The center of $\mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$ is generated by the endomorphisms $z_1,\dots,z_n$.*

Proof. By Corollary 3.3 we know already that $Z(H_d^{\boldsymbol{q}})$ is generated by the power sums $x_1^r + \cdots + x_d^r$ for $1 \leq r \leq d$. By Theorem 5.6, we can identify the endomorphism algebra $\mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$ with $eH_d^{\boldsymbol{q}}$ for some central idempotent $e \in H_d^{\boldsymbol{q}}$. So its center is generated by the restrictions of these power sums to the module $P^\mu \otimes V^{\otimes d}$. Now apply Corollary 5.3. $\quad\square$

Now let us restrict attention to a single block. For the rest of the article, we fix $\nu \in Y_d^\mu$ and define $\boldsymbol{i} = (i_1,\dots,i_d) \in X_d^{\boldsymbol{q}}$ from $\nu = \gamma - (\gamma_{i_1} - \gamma_{i_1+1}) - \cdots - (\gamma_{i_d} - \gamma_{i_d+1})$ as in Lemma 5.4. Let $e_\nu^\mu \in \mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$ be the central idempotent projecting $P^\mu \otimes V^{\otimes d}$ onto its component of generalized central character parametrized by ν. Note $(P^\mu \otimes V^{\otimes d})e_\nu^\mu$ is non-zero; see e.g. [BK, Lemma 4.2]. Identifying $\mathrm{End}_\mathfrak{g}(P^\mu \otimes V^{\otimes d})^{\mathrm{op}}$ with $eH_d^{\boldsymbol{q}}$ according to Theorem 5.6, it follows that e_ν^μ is identified with a non-zero central idempotent in $H_d^{\boldsymbol{q}}$.

Lemma 5.8. $e_\nu^\mu = b(\boldsymbol{i})$.

Proof. We first prove this in the special case that μ is a partition. Certainly e_ν^μ is a non-zero sum of the primitive central idempotents $b(\boldsymbol{i})$ for $\boldsymbol{i} \in X_d^{\boldsymbol{q}}$. So we just need to show that e_ν^μ acts as zero on $S_\lambda^{\boldsymbol{q}}$ for all $\boldsymbol{\lambda} \in \mathcal{M}_d(l)$

with $\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} \not\sim \boldsymbol{i}$. Given such a $\boldsymbol{\lambda}$, this is clear from the definition of the idempotent e in Theorem 5.6 unless $\ell(\lambda^{(r)}) \leq \mu_r$ for each $r = 1, \ldots, l$. In that case, there is a unique weight $\alpha \in \mathrm{Col}_d^\mu$ mapping to $\boldsymbol{\lambda}$ under the second bijection from Lemma 5.4, and the assumption that $\boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} \not\sim \boldsymbol{i}$ is equivalent to the statement that $\nu(\alpha) \neq \nu$. Let $N(\alpha)$ be the parabolic Verma module in $\mathcal{O}^\mu$ of highest weight $(\alpha - \rho)$. By [BK, Theorem 6.12], we have that

$$\mathrm{Hom}_{\mathfrak{g}}(P^\mu \otimes V^{\otimes d}, N(\alpha)) \cong S_{\boldsymbol{\lambda}}^{\boldsymbol{q}}$$

as $H_d^{\boldsymbol{q}}$-modules. Since $N(\alpha)$ and $(P^\mu \otimes V^{\otimes d})e_\nu^\mu$ have different generalized central characters, we have that

$$e_\nu^\mu \mathrm{Hom}_{\mathfrak{g}}(P^\mu \otimes V^{\otimes d}, N(\alpha)) = \mathrm{Hom}_{\mathfrak{g}}((P^\mu \otimes V^{\otimes d})e_\nu^\mu, N(\alpha)) = 0.$$

Hence $e_\nu^\mu S_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = 0$ as required.

To deduce the general case, assume still that μ is a partition and take another composition $\mu' \sim \mu$. We can find a central element $z_\nu \in Z(\mathfrak{g})$ that acts on $P^\mu \otimes V^{\otimes d}$ in the same way as e_ν^μ and on $P^{\mu'} \otimes V^{\otimes d}$ in the same way as $e_\nu^{\mu'}$. In the notation of Lemma 5.5, we have shown that $\rho^\mu(b(\boldsymbol{i}))$ coincides with the endomorphism of $P^\mu \otimes V^{\otimes d}$ defined by left multiplication by z_ν. We need to prove that $\rho^{\mu'}(b(\boldsymbol{i}))$ does too. This follows because the map $\iota_{\mu,\mu'}$ commutes with the action of z_ν.   □

Corollary 5.9. *The map $\rho_\nu^\mu : b(\boldsymbol{i})H_d^{\boldsymbol{q}} \to \mathrm{End}_{\mathfrak{g}}((P^\mu \otimes V^{\otimes d})e_\nu^\mu)^{\mathrm{op}}$ induced by the right action of $H_d^{\boldsymbol{q}}$ on $P^\mu \otimes V^{\otimes d}$ is an isomorphism. Moreover, the center of this algebra is generated by the endomorphisms $z_1, \ldots, z_n$, and is of dimension equal to the number of isomorphism classes of irreducible modules in $\mathcal{O}_\nu^\mu$.*

Proof.   The first statement is immediate from Theorem 5.6 since $e_\nu^\mu = b(\boldsymbol{i})$. The fact that the center of $\mathrm{End}_{\mathfrak{g}}((P^\mu \otimes V^{\otimes d})e_\nu^\mu)^{\mathrm{op}}$ is generated by the endomorphisms $z_1, \ldots, z_n$ is immediate from Corollary 5.7. The center is isomorphic to $Z(b(\boldsymbol{i})H_d^{\boldsymbol{q}})$, which by Theorem 4.3 is of dimension equal to the size of the set $\{\boldsymbol{\lambda} \in \mathcal{M}_d(l) \mid \boldsymbol{i}_{\boldsymbol{\lambda}}^{\boldsymbol{q}} = \boldsymbol{i}\}$. By Lemma 5.4, this is the same as the size of the set Col_ν^μ, that is, the number of isomorphism classes of irreducible modules in $\mathcal{O}_\nu^\mu$.   □
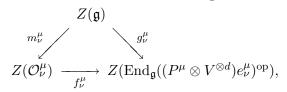
We need just one more fact, which is a variation on a result of Irving [I].

Lemma 5.10. *For any $\nu \in Y_d^\mu$, the injective hull of any module in $\mathcal{O}_\nu^\mu$ with a parabolic Verma flag is a finite direct sum of direct summands of $(P^\mu \otimes V^{\otimes d})e_\nu^\mu$.*

Proof.   We claim that every irreducible submodule of a parabolic Verma module in $\mathcal{O}_\nu^\mu$ embeds into $(P^\mu \otimes V^{\otimes d})e_\nu^\mu$. Since $(P^\mu \otimes V^{\otimes d})e_\nu^\mu$ is injective this implies the lemma. To prove the claim, recall that Y_d^μ is the set that parametrizes the central characters arising from irreducible constituents of $P^\mu \otimes V^{\otimes d}$. So the claim follows by [BK, Theorem 4.8] in the special case that μ is actually a partition. Essentially the same proof as there proves

the analogue of this theorem for arbitrary μ, providing one replaces the definition of "standard tableau" used in [BK] with the less familiar notion from [B1, (2.2)].   □

Now finally we consider the commutative diagram

$$Z(\mathfrak{g})$$

$$m_\nu^\mu \diagup \qquad \diagdown g_\nu^\mu$$

$$Z(\mathcal{O}_\nu^\mu) \xrightarrow{\ f_\nu^\mu\ } Z(\mathrm{End}_\mathfrak{g}((P^\mu \otimes V^{\otimes d})e_\nu^\mu)^{\mathrm{op}}),$$

where m_ν^μ, f_ν^μ and g_ν^μ are the natural multiplication maps.

Theorem 5.11. *In the above diagram, the maps m_ν^μ and g_ν^μ are surjective and the map f_ν^μ is an isomorphism. Hence, $Z(\mathcal{O}_\nu^\mu)$ is isomorphic to $Z(b(\boldsymbol{i})H_d^{\boldsymbol{q}})$ and is of dimension equal to the number of isomorphism classes of irreducible module in $\mathcal{O}_\nu^\mu$.*

Proof.   We first prove that f_ν^μ is injective. Suppose we are given a natural transformation $z \in Z(\mathcal{O}_\nu^\mu)$ defining the zero endomorphism of $(P^\mu \otimes V^{\otimes d})e_\nu^\mu$. To show that $z = 0$, we need to show that z defines the zero endomorphism of every module $M \in \mathcal{O}_\nu^\mu$. Let P be the projective cover of M and I be the injective hull of P. Since P has a parabolic Verma flag by general theory, Lemma 5.10 implies that I is a finite direct sum of summands of $(P^\mu \otimes V^{\otimes d})e_\nu^\mu$. Hence z defines the zero endomorphism of I. Since P embeds into I and surjects onto M, we get from this that z defines the zero endomorphism of M too. Now to finish the proof of the theorem, we know already from Corollary 5.9 that g_ν^μ is surjective. Hence by the commutativity of the diagram, m_ν^μ and f_ν^μ must both be surjetive too. The remaining statements are immediate from Corollary 5.9.   □

Finally, we note for any $\nu \in Y^\mu$ that tensoring with a sufficiently large power of determinant induces an equivalence between $\mathcal{O}_\nu^\mu$ and $\mathcal{O}_{\nu'}^\mu$ for some $\nu' \in Y_d^\mu$ and some $d \geq 0$. Given this, Theorem 2 from the introduction follows from Theorem 5.11.

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